

Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces

Xinru Cao*

School of Mathematical Science, Dalian University of Technology,
116023 Dalian, China

and

Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

May 27, 2014

Abstract In this paper, the fully parabolic Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & (x, t) \in \Omega \times (0, T), \\ v_t = \Delta v - v + u, & (x, t) \in \Omega \times (0, T), \end{cases} \quad (\star) \quad (0.1)$$

is considered under Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $n \geq 2$. We derive a smallness condition on the initial data in optimal Lebesgue spaces which ensure global boundedness and large time convergence. More precisely, we shall show that one can find $\varepsilon_0 > 0$ such that for all suitably regular initial data (u_0, v_0) satisfying $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} < \varepsilon_0$ and $\|\nabla v_0\|_{L^n(\Omega)} < \varepsilon_0$, the above problem possesses a global classical solution which is bounded and converges to the constant steady state (m, m) with $m := \frac{1}{|\Omega|} \int_{\Omega} u_0$.

Our approach allows us to furthermore study a general chemotaxis system with rotational sensitivity in dimension 2, which is lacking the natural energy structure associated with (\star) . For such systems, we prove a global existence and boundedness result under corresponding smallness conditions on the initially present total mass of cells and the chemical gradient.

1 Introduction

In this paper, we consider the initial-boundary value problem for two coupled parabolic equations,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u S(u, v, x) \cdot \nabla v), & (x, t) \in \Omega \times (0, T), \\ v_t = \Delta v - v + u, & (x, t) \in \Omega \times (0, T), \\ \nabla v \cdot \nu = (\nabla u - u S(x, u, v) \cdot \nabla v) \cdot \nu = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

*caoxinru@gmail.com

where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded domain with smooth boundary and ν is the normal outer vector on $\partial\Omega$. S is either a scalar number or $S(u, v, x) = (s_{i,j}(u, v, x))_{n \times n}$ is supposed to be a matrix with $s_{i,j} \in C^2([0, \infty) \times [0, \infty) \times \bar{\Omega})$ ($i, j = 1, 2$) and satisfies

$$|S(u, v, x)| \leq C_S \text{ with } C_S > 0. \quad (1.2)$$

Systems of this type describe the evolution of cell populations and their movement affected by the gradients of a chemical signal produced by the cells themselves, a mechanism commonly called chemotaxis. A classical chemotaxis system was proposed by Keller and Segel [10]. In the simplest form of their model, the first equation in (1.1) reads

$$u_t = \Delta u - \nabla \cdot (\chi u \nabla v), \quad (1.3)$$

where u, v denote the density of the cell population and chemical substance concentration, respectively. The number $\chi \in \mathbb{R}$ measures the sensitivity of the chemotactic response to the chemical gradients, and the second term on the right of (1.3) reflects the hypothesis that cells move towards *higher* densities of the signal. The second equation in (1.1) models the assumptions that the chemical substance is produced by cells and degrades. This kind of model has been widely studied during the last 40 years [15, 6]. We also refer to the survey [5, 4] for a broad overview.

Among the large quantity of the related researches, deciding whether solutions exist globally or blow up in finite time seems to be one of the most challenging mathematical topics [18]. In the two-dimensional setting, a critical mass phenomenon has been identified and studied in many works. In the case $\int_{\Omega} u_0 < 4\pi$, the solution is global and bounded [11], whereas if $\int_{\Omega} u_0 > 4\pi$, the occurrence of blow-up for some initial data is only detected when the second equation is replaced by an elliptic equation of the form $-\Delta v + v - u = 0$ or $-\Delta v - (u - \bar{u}_0) = 0$, which reflects a certain limit procedure [7]. In higher dimensions, there are many results for such simplified parabolic-elliptic versions. For instance, in [2], it is proved that the corresponding Cauchy problem possesses a global weak solution whenever $\|u_0\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} < C$ with some $C > 0$. Considering the fully parabolic case, it is also shown that for small initial data,

$$\|u_0\|_{L^q(\mathbb{R}^n)} < \varepsilon, \text{ and } \|\nabla v_0\|_{L^p(\mathbb{R}^n)} < \varepsilon \quad (1.4)$$

with some suitably small constant $\varepsilon > 0$, $q > \frac{n}{2}$ and $p \geq n$, the solution exists globally and is bounded [3]. However, for the fully parabolic version in bounded domains, the same conclusion is up to now known to hold only for $q > \frac{n}{2}$ and $p > n$ [17]. It is our goal to extend this result in the corresponding critical case, that is, for $q = \frac{n}{2}$ and $p = n$.

In contrast to (1.3), a recent study suggests a more general model which allows a wider direction of the cells' movement, such as they move not to the higher density of chemical any more but with a rotation. Then in this system a sensitivity *tensor* $S(u, v, x)$, instead of a *scalar* constant χ , is introduced to describe chemotactic motion [14]. The introduction of this *tensor* valued sensitivity is caused by a kind of complicated interactions between the cell motion speed and directional effects stemming from the action of gravity, for example.

In our study of this new model, we concentrate on the two-dimensional case, and we anticipate that small mass of u guarantees global existence, which indeed parallels the case of a scalar sensitivity. However, the classical way of proof in the scalar case strongly depends on the use of an energy inequality [11], which is apparently lacking in the general system. To the best of our knowledge, the only results

on global existence and boundedness in a related case can be found in [9], but with the second equation being replaced by $v_t = \Delta v - f(u)v$, whereby v enjoys an a priori upper bound according to the obvious estimate $v \leq \|v_0\|_{L^\infty(\Omega)}$. Under mild assumptions on S and f , the authors in that work proved that the solution exists globally and is bounded if either $\|v_0\|_{L^\infty(\Omega)}$ or $\|u_0\|_{L^1(\Omega)}$ is small enough.

Since the second equation of (1.1) has a production term, the method in [9] does not apply to the present situation. However, we may benefit from our approach developed above for the case of scalar sensitivities in order to prove global boundedness under the assumptions that (1.2) holds, and that both $\|u_0\|_{L^1(\Omega)}$ and $\|\nabla v_0\|_{L^2(\Omega)}$ are small enough. We underline that this assumption on the initial data is still stronger than that in the case of scalar sensitivity, but our results include exponential convergence, which has not been found before without assuming $\|\nabla v_0\|_{L^2(\Omega)}$ small enough.

Our main result says that

- if $n \geq 2$, one can find an upper bound for u_0 in $L^{\frac{n}{2}}(\Omega)$ and ∇v_0 in $L^n(\Omega)$, such that the solution (u, v) of (1.1) exists globally and is bounded.

2 Preliminaries

In this section, we recall some classical $L^p - L^q$ estimates for the Neumann heat semigroup on bounded domains. Almost all of the results and their proofs can be found in [17, Lemma 1.3]. However, some of the estimates we use below go slightly beyond, and since we could not find a precise reference, we will give a short proof here.

Lemma 2.1 *Suppose $(e^{t\Delta})_{t>0}$ is the Neumann heat semigroup in Ω , and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist $k_1, \dots, k_4 > 0$ which only depend on Ω and which have the following properties:*

- (i) *If $1 \leq q \leq p \leq \infty$, then*

$$\|e^{t\Delta}w\|_{L^p(\Omega)} \leq k_1(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \text{ for all } t > 0 \quad (2.1)$$

holds for all $w \in L^q(\Omega)$ with $\int_\Omega w = 0$.

- (ii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|\nabla e^{t\Delta}w\|_{L^p(\Omega)} \leq k_2(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \text{ for all } t > 0 \quad (2.2)$$

holds for each $w \in L^q(\Omega)$.

- (iii) *If $2 \leq q \leq p < \infty$, then*

$$\|\nabla e^{t\Delta}w\|_{L^p(\Omega)} \leq k_3 e^{-\lambda_1 t}(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})\|\nabla w\|_{L^q(\Omega)} \text{ for all } t > 0 \quad (2.3)$$

is true for all $w \in W^{1,p}(\Omega)$.

- (iv) *Let $1 < q \leq p \leq \infty$, then*

$$\|e^{t\Delta}\nabla \cdot w\|_{L^p(\Omega)} \leq k_4(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \text{ for all } t > 0 \quad (2.4)$$

is valid for any $w \in (W^{1,p}(\Omega))^n$.

Proof (i) and (ii) are precisely proved in [17, Lemma 1.3]. Focusing on (iii), we note that it is obviously true for all $t < 2$ [17]. If $t \geq 2$, let $\bar{w} = \frac{1}{|\Omega|} \int_{\Omega} w$, we use (2.1), (2.2) and the Poincaré inequality to obtain

$$\begin{aligned} \|\nabla e^{t\Delta} w\|_{L^p(\Omega)} &= \|\nabla e^{\Delta} e^{(t-1)\Delta} (w - \bar{w})\|_{L^p(\Omega)} \\ &\leq 2k_2 \|e^{(t-1)\Delta} (w - \bar{w})\|_{L^p(\Omega)} \\ &\leq 2k_2 \cdot k_1 (1 + (t-1)^{-\frac{2}{q}(\frac{1}{q}-\frac{1}{p})}) e^{-\lambda_1(t-1)} \|w - \bar{w}\|_{L^q(\Omega)} \\ &\leq k_3 (1 + t^{-\frac{2}{q}(\frac{1}{q}-\frac{1}{p})}) e^{-\lambda_1 t} \|\nabla w\|_{L^q(\Omega)}. \end{aligned}$$

Thus (iii) is valid for all $t > 0$. Note that k_3 is independent of p or q since the constant in Poincaré inequality could be independent of q .

In [17], (2.4) is proved for $1 < q \leq p < \infty$, so that it is sufficient to prove the case $1 < q < p = \infty$. Suppose that $w \in (C_0^\infty(\Omega))^n$. Then $\int_{\Omega} e^{\frac{t}{2}\Delta} \nabla \cdot w = \int_{\Omega} \nabla \cdot w = 0$, whence from (2.1) we see that

$$\begin{aligned} \|e^{t\Delta} \nabla \cdot w\|_{L^\infty(\Omega)} &= \|e^{\frac{t}{2}\Delta} (e^{\frac{t}{2}\Delta} \nabla \cdot w)\|_{L^\infty(\Omega)} \leq k_1 (1 + (\frac{t}{2})^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{\infty})}) e^{-\lambda_1 \frac{t}{2}} \|e^{\frac{t}{2}\Delta} \nabla \cdot w\|_{L^q(\Omega)} \\ &\leq k_1 (1 + (\frac{t}{2})^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{\infty})}) e^{-\lambda_1 \frac{t}{2}} \cdot k_2 (1 + (\frac{t}{2})^{-\frac{1}{2}}) e^{-\lambda_1 \frac{t}{2}} \|w\|_{L^q(\Omega)} \\ &\leq k_3 (1 + t^{-\frac{1}{2}-\frac{n}{2q}}) e^{-\lambda_1 t} \|w\|_{L^q(\Omega)} \end{aligned}$$

for all $t \in (0, T)$ with $k_3 = 3k_1 k_2$. Thus (2.4) is obtained by means of a unique extension to all of $(W^{1,p}(\Omega))^n$. \square

Before going into the main part, we also recall some local existence and extensibility results for (1.1). We refer to [1, Propersition 2.3] for the case with rotation. See also [17, Lemma 1.1] for the case where S is a scalar constant. Beyond these, we generalize as follows.

Lemma 2.2 *Suppose that either $S(x, u, v) \in \mathbb{R}$ or $S(x, u, v)$ is a matrix satisfying (1.2) and in addition that $S(x, u, v) = 0$ on $\partial\Omega$. Assume $u_0 \in C(\bar{\Omega})$ and $v_0 \in W^{1,\sigma}(\Omega)$ are nonnegative with $\sigma > n$. Then there exists $T_{\max} > 0$ such that (1.1) possesses a classical solution $(u, v) \in (C^0([0, T_{\max}) \times \bar{\Omega}) \cap C^{2,1}((0, T_{\max}) \times \bar{\Omega}))^2$. Moreover, $T_{\max} < \infty$ if and only if*

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (2.5)$$

Now we give a sufficient condition for boundedness and global existence.

Lemma 2.3 *Let $\theta > \frac{n}{2}$. If the solution of (1.1) satisfies*

$$\|u\|_{L^\theta(\Omega)} < \infty \text{ for all } t \in (0, T_{\max}), \quad (2.6)$$

then $T_{\max} = \infty$, and $\sup_{t>0} (\|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)}) < \infty$.

The following lemma presents an estimate for certain integrals, which we will frequently use in the next section. The proof can be found in [17, Lemma 1.2].

Lemma 2.4 *There exists $C > 0$ such that for $\alpha < 1$, $\beta < 1$ and γ, δ be positive and satisfy $\gamma \neq \delta$. Then*

$$\int_0^t (1 + (t-s)^{-\alpha}) e^{-\gamma(t-s)} (1 + s^{-\beta}) e^{-\delta s} ds \leq C(\alpha, \beta, \delta, \gamma) (1 + t^{\min\{0, 1-\alpha-\beta\}}) e^{-\min\{\gamma, \delta\} \cdot t} \quad (2.7)$$

for all $t > 0$, where $C(\alpha, \beta, \delta, \gamma) := 2C \cdot (\frac{1}{|\delta-\gamma|} + \frac{1}{1-\alpha} + \frac{1}{1-\beta})$.

3 Smallness conditions in optimal spaces

Now having at hand the tools collected in the last section, we can prove global existence in the classical Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & (x, t) \in \Omega \times (0, T), \\ v_t = \Delta v - v + u, & (x, t) \in \Omega \times (0, T), \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (3.1)$$

where $\Omega \subset \mathbb{R}^n$ with smooth boundary and $n \geq 2$.

Our proof relies on a fixed-point type argument developed in [17]. Unlike in the original proof there, our assumption is that the initial data be suitably small in *optimal* spaces in the sense that we require that

$$\|u_0\|_{L^{\frac{n}{2}}(\Omega)} \text{ and } \|\nabla v_0\|_{L^n(\Omega)} \text{ are small.}$$

This seems too weak to give an $L^\infty(\Omega)$ bound in a one-step procedure. However, we can first use the "weak" assumption to obtain the smallness condition in supercritical spaces, which meet the assumptions in [17, Theorem 2.1]. Finally, we can derive convergence in $L^\infty(\Omega)$, and obtain the convergence rate $e^{-\lambda' t}$ with any $0 < \lambda' < \lambda_1$, where $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions.

In Theorem 1, we show how we improve the smallness condition into a supercritical space.

Theorem 1 *Let $n \geq 2$, $0 < \lambda' < \lambda_1$. Then there exists $\varepsilon_0 > 0$ depending on λ' and Ω with the following property: If $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\sigma}(\Omega)$ with $\sigma > n$ are nonnegative and satisfy*

$$\|u_0\|_{L^{\frac{n}{2}}(\Omega)} \leq \varepsilon \text{ and } \|\nabla v_0\|_{L^n(\Omega)} \leq \varepsilon \quad (3.2)$$

for some $\varepsilon < \varepsilon_0$, then (3.1) possesses a global classical solution (u, v) which is bounded and satisfies

$$\|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\theta(\Omega)} \leq \varepsilon(1 + t^{-1+\frac{n}{2\theta}})e^{-\lambda' t} \text{ for all } t > 0, \quad (3.3)$$

for all $\theta \in [q_0, \theta_0]$, where $\frac{n}{2} < q_0 < n$ and $n < \theta_0 < \frac{nq_0}{n-q_0}$.

Proof First we fix $0 < \lambda' < \lambda_1$. Since $\frac{n}{2} < q_0 < n$ and $n < \theta_0 < \frac{nq_0}{n-q_0}$, it is possible to fix $q_1, q_2 > 0$ satisfying $q_0 < q_1 < \theta_0$, $n < q_2 < \frac{nq_0}{n-q_0}$ and $\frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2}$. By applying Lemma 2.4, we can find constants $c_1, c_2, c_3 > 0$ such that for all $\theta \in [q_0, \theta_0]$,

$$\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{p})})e^{-(\lambda_1+1)(t-s)}(1 + s^{-1+\frac{n}{2\theta}})e^{-\lambda' s} ds \leq c_1(1 + t^{\min\{0, -\frac{1}{2}+\frac{n}{2\theta}\}})e^{-\lambda' t} \quad (3.4)$$

$$\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)}(1 + s^{-\frac{3}{2}+\frac{n}{2q_0}})e^{-\lambda' s} ds \leq c_2(1 + t^{\min\{0, -1+\frac{n}{2\theta}\}})e^{-\lambda' t} \quad (3.5)$$

$$\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})})e^{-\lambda_1(t-s)}(1 + s^{-\frac{1}{2}+\frac{n}{2q_0}})e^{-\lambda' s} ds \leq c_3(1 + t^{\min\{0, \frac{n}{2\theta}\}})e^{-\lambda' t} \quad (3.6)$$

hold for all $t > 0$. Since $p < \frac{nq_0}{n-q_0}$, we know that $\frac{1}{2} - \frac{n}{2}(\frac{1}{\theta_0} - \frac{1}{p}) > \frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta_0}) > 0$, and $\lambda_1 + 1 - \lambda' > 1$, then we see that $c_1 > 0$ is dependent on Ω only. Similarly, it is not difficult to derive that $\frac{1}{2} - \frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta}) \geq \frac{1}{2} - \frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta_0}) > 0$ and $1 - 1 + \frac{n}{2q_1} - \frac{1}{2} + \frac{n}{2q_2} = -\frac{1}{2} + \frac{n}{2q_0} > 0$ due to $q_0 < n$ and $\theta_0 < \frac{nq_0}{n-q_0}$. Therefore,

$c_2 > 0$ depend only on λ', Ω . More precisely, $c_2 \rightarrow \infty$ as $\lambda' \rightarrow \lambda_1$. Noting that $1 - \frac{1}{2} + \frac{n}{2q_0} > \frac{1}{2} > 0$, c_3 depends on λ' and Ω .

With $\varepsilon_0 > 0$ to be determined below, we assume (3.2) holds for $\varepsilon \in (0, \varepsilon_0)$, and define

$$T := \sup \left\{ \tilde{T} > 0 \mid \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\theta(\Omega)} \leq \varepsilon(1 + t^{-1+\frac{n}{2\theta}})e^{-\lambda't}, \text{ for all } t \in [0, \tilde{T}), \text{ for all } \theta \in [q_0, \theta_0] \right\}. \quad (3.7)$$

We observe that T is well defined and positive due to Lemma 2.2. It is sufficient to prove that $T = \infty$. Since $n \geq 2$, it is easy to see that

$$\bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0 \leq |\Omega|^{-\frac{2}{n}} \|u_0\|_{L^{\frac{n}{2}}(\Omega)} \leq |\Omega|^{-\frac{2}{n}} \varepsilon. \quad (3.8)$$

Then by the definition of T and (2.1), for each $\theta \in [q_0, \theta_0]$,

$$\begin{aligned} \|u(\cdot, t) - \bar{u}_0\|_{L^\theta(\Omega)} &\leq \|u - e^{t\Delta} u_0\|_{L^\theta(\Omega)} + \|e^{t\Delta} u_0 - \bar{u}_0\|_{L^\theta(\Omega)} \\ &\leq \varepsilon(1 + t^{-1+\frac{n}{2\theta}})e^{-\lambda't} + k_1(1 + t^{-1+\frac{n}{2\theta}})e^{-\lambda_1 t} \|u_0 - \bar{u}_0\|_{L^{\frac{n}{2}}(\Omega)} \\ &\leq (\varepsilon + k_1 \|u_0\|_{L^{\frac{n}{2}}(\Omega)} + k_1 \bar{u}_0 |\Omega|^{\frac{2}{n}})(1 + t^{-1+\frac{n}{2\theta}})e^{-\lambda't} \\ &\leq c_4 \varepsilon (1 + t^{-1+\frac{n}{2\theta}})e^{-\lambda't} \end{aligned} \quad (3.9)$$

holds for all $t \in [0, T)$, where $c_4 = 1 + 2k_1$ only depends on Ω .

Next, we claim that there exists $c_5 > 0$ such that for all $p \in [q_0, \frac{nq_0}{n-q_0})$,

$$\|\nabla(v(\cdot, t) - e^{t(\Delta-1)} v_0)\|_{L^p(\Omega)} \leq c_5 \varepsilon (1 + t^{-\frac{1}{2}+\frac{n}{2p}})e^{-\lambda't} \text{ for all } t \in [0, T), \quad (3.10)$$

where $c_5 > 0$ depends only on Ω . Indeed, by using the variation-of-constants formula for v ,

$$v(\cdot, t) = e^{t(\Delta-1)} v_0 + \int_0^t e^{(t-s)(\Delta-1)} u(\cdot, s) ds \quad \text{for all } t \in (0, T)$$

and with the fact that $\nabla e^{t\Delta} \bar{u}_0 = \nabla \bar{u}_0 = 0$, we obtain

$$\|\nabla(v(\cdot, t) - e^{t(\Delta-1)} v_0)\|_{L^p(\Omega)} \leq \int_0^t \|\nabla e^{(t-s)(\Delta-1)} (u(\cdot, s) - \bar{u}_0)\|_{L^p(\Omega)} ds$$

for all $t \in [0, T)$. We observe that (2.2) (3.9) and (3.4) imply

$$\begin{aligned} &\|\nabla(v(\cdot, t) - e^{t(\Delta-1)} v_0)\|_{L^p(\Omega)} \\ &\leq \int_0^t \|\nabla e^{(t-s)(\Delta-1)} (u(\cdot, s) - \bar{u}_0)\|_{L^p(\Omega)} ds \\ &\leq \int_0^t k_2 (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{p})}) e^{-(\lambda_1+1)(t-s)} \|u - \bar{u}_0\|_{L^\theta(\Omega)} ds \\ &\leq \int_0^t k_2 (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{p})}) e^{-(\lambda_1+1)(t-s)} c_4 \varepsilon (1 + s^{-1+\frac{n}{2\theta}}) e^{-\lambda's} ds \\ &\leq k_2 c_4 c_1 \varepsilon (1 + t^{-\frac{1}{2}+\frac{n}{2p}}) e^{-\lambda't} \end{aligned}$$

for all $t \in [0, T)$ and all $p \in [q_0, \frac{nq_0}{n-q_0})$. Now we gain (3.10) by letting $c_5 = k_2 c_1 c_4$, where $c_5 > 0$ depends on Ω only.

On the other hand, by applying (2.3) and making use of (3.2), we obtain

$$\|\nabla e^{t(\Delta-1)} v_0\|_{L^p(\Omega)} \leq k_3 e^{-(\lambda_1+1)t} (1 + t^{-\frac{1}{2}+\frac{n}{2p}}) \|\nabla v_0\|_{L^n(\Omega)} \leq k_3 \varepsilon (1 + t^{-\frac{1}{2}+\frac{n}{2p}}) e^{-(\lambda_1+1)t} \quad (3.11)$$

for all $t \in [0, T)$ and $p \in [n, \frac{nq_0}{n-q_0})$. We observe that a slightly modified version of (3.11) is also true for $p < n$, because Hölder's inequality, (2.2) and (3.2) yield

$$\begin{aligned} \|\nabla e^{t(\Delta-1)} v_0\|_{L^p(\Omega)} &\leq |\Omega|^{\frac{1}{p}-\frac{1}{n}} \|\nabla e^{t(\Delta-1)} v_0\|_{L^n(\Omega)} \leq k_2 \max\{1, |\Omega|, |\Omega|^{-1}\} e^{-(\lambda_1+1)t} \|\nabla v_0\|_{L^n(\Omega)} \\ &\leq c_6 \varepsilon e^{-(\lambda_1+1)t} \leq c_6 \varepsilon (1+t^{-\frac{1}{2}+\frac{n}{2p}}) e^{-\lambda' t} \end{aligned} \quad (3.12)$$

for all $t \in (0, T)$ with $c_6 = k_2 \max\{1, |\Omega|\}$ independent of p . Thus collecting (3.10), (3.11) and (3.12), and observing that $\lambda_1 + 1 > \lambda'$, we conclude that there exists $c_7 > 0$ such that

$$\begin{aligned} \|\nabla v(\cdot, t)\|_{L^p(\Omega)} &\leq c_5 \varepsilon (1+t^{-\frac{1}{2}+\frac{n}{2p}}) e^{-\lambda' t} + (c_6 + k_3) \varepsilon (1+t^{-\frac{1}{2}+\frac{n}{2p}}) e^{-(\lambda_1+1)t} \\ &\leq c_7 \varepsilon (1+t^{-\frac{1}{2}+\frac{n}{2p}}) e^{-\lambda' t} \end{aligned} \quad (3.13)$$

for all $t \in [0, T)$ and all $p \in [q_0, \frac{nq_0}{n-q_0})$, where $c_7 = c_5 + c_6 + k_3$ depends on Ω only.

Now we use the variation-of-constants formula associated with u to infer that

$$\begin{aligned} \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\theta(\Omega)} &\leq \int_0^t \|e^{(t-s)\Delta} \nabla \cdot (u \nabla v)\|_{L^\theta(\Omega)} ds \\ &\leq \int_0^t k_4 (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) e^{-\lambda_1(t-s)} \|u(\cdot, s) \nabla v(\cdot, s)\|_{L^{q_0}(\Omega)} ds \\ &\leq \int_0^t k_4 (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) e^{-\lambda_1(t-s)} \|(u - \bar{u}_0) \nabla v(\cdot, s)\|_{L^{q_0}(\Omega)} ds \\ &\quad + \int_0^t k_4 (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) e^{-\lambda_1(t-s)} \bar{u}_0 \|\nabla v(\cdot, s)\|_{L^{q_0}(\Omega)} ds \\ &=: I_1 + I_2 \end{aligned} \quad (3.14)$$

for all $t \in [0, T)$, where k_4 from (2.4) depends on Ω only. To estimate I_1 , we recall the choices of q_1 and q_2 , which ensure that $-1 + \frac{n}{2q_1} < 0$, $-\frac{1}{2} + \frac{n}{2q_2} < 0$ and $-1 + \frac{n}{2q_1} - \frac{1}{2} + \frac{n}{2q_2} = -\frac{3}{2} + \frac{n}{2q_0} > -1$ as well as $-\frac{1}{2} - \frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta}) > -\frac{1}{2} - \frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta_0}) > -1$. Moreover, since $q_0 < q_1 < \theta_0$ and $n < q_2 < \frac{nq_0}{n-q_0}$, in light of the Hölder inequality, (3.9) (3.13) and (3.5), we conclude for all $\theta \in [q_0, \theta_0]$ that

$$\begin{aligned} I_1 &\leq \int_0^t k_4 (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) e^{-\lambda_1(t-s)} \|u(\cdot, s) - \bar{u}_0\|_{L^{q_1}(\Omega)} \|\nabla v(\cdot, s)\|_{L^{q_2}(\Omega)} ds \\ &\leq \int_0^t k_4 (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) e^{-\lambda_1(t-s)} c_4 \varepsilon (1+s^{-1+\frac{n}{2q_1}}) e^{-\lambda' s} c_7 \varepsilon (1+s^{-\frac{1}{2}+\frac{n}{2q_2}}) e^{-\lambda' s} ds \\ &\leq \int_0^t 4c_4 c_7 k_4 \varepsilon^2 (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) e^{-\lambda_1(t-s)} (1+s^{-1+\frac{n}{2q_1}-\frac{1}{2}+\frac{n}{2q_2}}) e^{-\lambda' s} ds \\ &\leq 4c_4 c_7 k_4 c_2 \varepsilon^2 (1+t^{\min\{0, 1-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})-1+\frac{n}{2q_1}-\frac{1}{2}+\frac{n}{2q_2}\}}) e^{-\lambda' t} \\ &=: c_8 \varepsilon^2 (1+t^{-1+\frac{n}{2\theta}}) e^{-\lambda' t} \end{aligned} \quad (3.15)$$

for all $t \in [0, T)$ with $c_8 = 4c_4 c_7 k_4 c_2$. We see that $c_8 > 0$ depend only on λ', Ω . More precisely, $c_8 \rightarrow \infty$ as $\lambda' \rightarrow \lambda_1$.

Similarly, (3.13) together with (3.6) imply

$$\begin{aligned} I_2 &\leq \int_0^t k_4 (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) e^{-\lambda_1(t-s)} \bar{u}_0 \|\nabla v(\cdot, s)\|_{L^{q_0}(\Omega)} ds \\ &\leq \int_0^t k_4 (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q_0}-\frac{1}{\theta})}) e^{-\lambda_1(t-s)} |\Omega|^{-\frac{2}{n}} \varepsilon \cdot c_7 \varepsilon (1+s^{-\frac{1}{2}+\frac{n}{2q_0}}) e^{-\lambda' s} ds \end{aligned}$$

$$\begin{aligned}
&\leq c_7 k_4 c_3 |\Omega|^{-\frac{2}{n}} \varepsilon^2 (1 + t^{\min\{0, 1 - \frac{1}{2} - \frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta}) - \frac{1}{2} + \frac{n}{2q_0}\}}) e^{-\lambda' t} \\
&\leq c_7 k_4 c_3 |\Omega|^{-\frac{2}{n}} \varepsilon^2 (1 + t^{\min\{0, \frac{n}{2\theta}\}}) e^{-\lambda' t} \\
&\leq 2c_7 k_4 c_3 |\Omega|^{-\frac{2}{n}} \varepsilon^2 e^{-\lambda' t} \leq c_9 \varepsilon^2 (1 + t^{-1 + \frac{n}{2\theta}}) e^{-\lambda' t}
\end{aligned} \tag{3.16}$$

for all $t \in [0, T)$ with $c_9 = 2c_7 k_4 c_3 |\Omega|^{-\frac{2}{n}}$. Thus c_9 depends on Ω and λ' , and $c_9 \rightarrow \infty$ as $\lambda' \rightarrow \lambda_1$. As a consequence of (3.14), (3.15) and (3.16), we arrive at

$$\|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\theta(\Omega)} \leq c_{10} \varepsilon^2 (1 + t^{-1 + \frac{n}{2\theta}}) e^{-\lambda' t} \text{ for all } t \in [0, T), \theta \in [q_0, \theta_0] \tag{3.17}$$

where $c_{10} = c_8 + c_9$ depends only on λ' and Ω . Choosing $\varepsilon_0 < \frac{1}{2c_{10}}$, we conclude that $T = \infty$.

Since $T \leq T_{\max}$, we conclude that $T_{\max} = \infty$. In order to prove (u, v) is bounded, we first obtain from the definition of T and (3.8) that

$$\|u(\cdot, t)\|_{L^\theta(\Omega)} \leq \|u(\cdot, t) - \bar{u}_0\|_{L^\theta} + \bar{u}_0 |\Omega|^{\frac{1}{\theta}} \leq c_4 \varepsilon (1 + t^{-1 + \frac{n}{2\theta}}) e^{-\lambda' t} + |\Omega|^{\frac{1}{\theta} - \frac{2}{n}} \varepsilon \tag{3.18}$$

is true for $t \in [1, \infty)$ for any choice of $\theta \in [q_0, \theta_0)$. The result of local existence yields

$$\|u(\cdot, t)\|_{L^\theta(\Omega)} < \infty \tag{3.19}$$

for all $t < 1$. We see that (3.19) together with (3.18) yields that there exists $c_{11} > 0$ such that

$$\|u(\cdot, t)\|_{L^\theta(\Omega)} \leq c_{11} \tag{3.20}$$

for all $\theta \in [q_0, \theta_0)$ and all $t > 0$. Since $\theta \geq q_0 > \frac{n}{2}$, (3.20) implies boundedness of (u, v) by Lemma 2.3. Thus (u, v) is global and bounded. \square

Remark 3.1 A careful re-inspection of the above argument shows that for the constant $c_{10} = c_{10}(\lambda')$ satisfies $c_{10}(\lambda') \rightarrow \infty$ as $\lambda' \rightarrow \lambda_1$, where the constant $\varepsilon_0 = \varepsilon_0(\lambda')$ by Theorem 1 has the property that $\varepsilon_0(\lambda') \rightarrow 0$ as $\lambda' \rightarrow \lambda_1$.

With the decay property in higher Lebesgue spaces obtained above, we can obtain a smallness condition which ensures stabilization of (u, v) in $L^\infty(\Omega)$.

Theorem 2 Let $n \geq 2$, $0 < \lambda' < \lambda_1$, then there exists $\hat{\varepsilon}_0 > 0$ with the following property:

If $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\sigma}(\Omega)$ with $\sigma > n$ are nonnegative and

$$\|u_0\|_{L^{\frac{n}{2}}(\Omega)} \leq \varepsilon \text{ and } \|\nabla v_0\|_{L^n(\Omega)} \leq \varepsilon \tag{3.21}$$

for some $\varepsilon \leq \hat{\varepsilon}_0$, then (3.1) possesses a global classical solution (u, v) which is bounded and satisfies

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} \leq \bar{C} e^{-\lambda' t} \text{ and } \|v(\cdot, t) - \bar{v}_0\|_{L^\infty(\Omega)} \leq \bar{C} e^{-\min\{\lambda', 1\}t} \text{ for all } t > 0, \tag{3.22}$$

where $\bar{C} > 0$ depends on λ' and Ω .

Proof First we fix $q_0 \in (\frac{n}{2}, n)$, $\theta_0 \in (n, \frac{nq_0}{n-q_0})$ and $\theta \in [q_0, \theta_0]$ as well as $p \in [q_0, \frac{nq_0}{n-q_0}]$. In particular, we know that $\theta > \frac{n}{2}$ and $p > n$. By applying Theorem 2.1 in [17], we see that there exists $\varepsilon_1 > 0$, such that for any nonnegative initial data $(\tilde{u}_0, \tilde{v}_0) \in C(\bar{\Omega}) \times W^{1,\sigma}(\Omega)$ with $\sigma > n$ satisfying

$$\|\tilde{u}_0\|_{L^\theta(\Omega)} \leq \varepsilon_1, \quad \|\nabla \tilde{v}_0\|_{L^p(\Omega)} \leq \varepsilon_1, \tag{3.23}$$

then the solution (\tilde{u}, \tilde{v}) of (3.1) with the initial data $(\tilde{u}_0, \tilde{v}_0)$ exists globally and satisfies

$$\begin{aligned} \|\tilde{u}(\cdot, t) - e^{t\Delta}\tilde{u}_0\|_{L^\infty(\Omega)} &\leq C\varepsilon^2 e^{-\lambda't}, \\ \|\tilde{u}(\cdot, t) - \bar{\tilde{u}}_0\|_{L^\infty} &\leq C e^{-\lambda't}, \text{ and } \|\tilde{v}(\cdot, t) - \bar{\tilde{u}}_0\|_{L^\infty(\Omega)} \leq C e^{-\min\{\lambda', 1\}t} \end{aligned} \quad (3.24)$$

for all $t > 0$ and some $C > 0$. According to (3.9) and (3.13), there exists $\varepsilon_0 > 0$ such that for any initial data satisfying (3.21),

$$\begin{aligned} \|u(\cdot, t)\|_{L^\theta(\Omega)} &\leq \|u(\cdot, t) - \bar{u}_0\|_{L^\theta(\Omega)} + \bar{u}_0|\Omega|^{\frac{1}{\theta}} \leq \varepsilon_0(1 + t^{-1+\frac{n}{2\theta}})e^{-\lambda't} + |\Omega|^{\frac{1}{\theta}-\frac{2}{n}}\varepsilon_0, \\ \|\nabla v(\cdot, t)\|_{L^p(\Omega)} &\leq c_7\varepsilon_0(1 + t^{-1+\frac{n}{2p}})e^{-\lambda't} \end{aligned}$$

hold for all $t > 0$. From this, we observe that there exists $t_0 > 0$ such that for all $t \geq t_0$,

$$\begin{aligned} \|u(\cdot, t)\|_{L^\theta(\Omega)} &\leq 2|\Omega|^{\frac{1}{\theta}-\frac{2}{n}}\varepsilon_0, \\ \|\nabla v(\cdot, t)\|_{L^p(\Omega)} &\leq 2|\Omega|^{\frac{1}{\theta}-\frac{2}{n}}\varepsilon_0. \end{aligned}$$

Let $\tilde{u}_0 = u(t_0)$, $\tilde{v}_0 = v(t_0)$, we easily see that $\bar{\tilde{u}}_0 = \bar{u}_0$, $\tilde{u}(t) = u(t + t_0)$ and $\tilde{v}(t) = v(t + t_0)$. Taking $\hat{\varepsilon}_0 = \min\{\varepsilon_0, \frac{1}{2}|\Omega|^{\frac{2}{n}-\frac{1}{\theta}}\varepsilon_1\}$ and substituting (u, v) into (3.24) will complete the proof. \square

4 System with rotational sensitivity

In this section, we consider the modified Keller-Segel system with general tensor-valued sensitivity as given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (uS(u, v, x) \cdot \nabla v), & (x, t) \in \Omega \times (0, T), \\ v_t = \Delta v - v + u, & (x, t) \in \Omega \times (0, T), \\ (\nabla u - uS(u, v, x) \cdot \nabla v) \cdot \nu = 0, \quad \nabla v \cdot \nu = 0 & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (4.1)$$

where $\Omega \subset \mathbb{R}^2$ with smooth boundary. The sensitivity S is now supposed to be a tensor-valued function satisfying (1.2). The non-flux and coupled boundary condition complicate the solvability. Following [9], we first regularize the system as below

$$\begin{cases} (u_\eta)_t = \Delta u_\eta - \nabla \cdot (u_\eta S_\eta(u_\eta, v_\eta, x) \cdot \nabla v_\eta), & (x, t) \in \Omega \times (0, T), \\ (v_\eta)_t = \Delta v_\eta - v_\eta + u_\eta, & (x, t) \in \Omega \times (0, T), \\ (\nabla u_\eta - u_\eta S_\eta(u_\eta, v_\eta, x) \cdot \nabla v_\eta) \cdot \nu = 0, \quad \nabla v_\eta \cdot \nu = 0 & (x, t) \in \partial\Omega \times (0, T), \\ u_\eta(x, 0) = u_0(x), \quad v_\eta(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (4.2)$$

with $S_\eta(x, u_\eta, v_\eta) = \rho_\eta(x)S(u_\eta, v_\eta, x)$, which vanishes on the boundary $\partial\Omega$ if ρ_η is a suitable cut-off function on Ω . Where $\rho_\eta \in [0, 1]$ and satisfies

$$\rho_\eta \rightarrow 1 \text{ a.e. as } \eta \rightarrow 0. \quad (4.3)$$

The first boundary condition of (4.1) is then reduced to $\nabla u_\eta \cdot \nu = 0$, so that local classical solvability of (4.1) can be obtained by the standard approach (Lemma2.2). Upon combining the idea in the previous section with a limiting procedure $\eta \rightarrow 0$, we will derive the following.

Theorem 3 *Let S satisfy (1.2) and $0 < \lambda' < \lambda_1$. Then there exists $\epsilon > 0$ with the property that if the nonnegative initial data $u_0 \in C(\bar{\Omega})$ and $v_0 \in W^{1,\sigma}(\Omega)$ satisfy*

$$\|u_0\|_{L^1(\Omega)} \leq \epsilon \text{ and } \|\nabla v_0\|_{L^2(\Omega)} \leq \epsilon \quad (4.4)$$

then (4.1) possesses a global classical solution (u, v) which is bounded and satisfies

$$\|u - \bar{u}_0\|_{L^\infty(\Omega)} \leq \bar{C}e^{-\lambda't}, \|v - \bar{u}_0\|_{L^\infty(\Omega)} \leq \bar{C}e^{-\min\{\lambda', 1\}t} \quad (4.5)$$

with some $\bar{C} > 0$.

Before we proceed to prove Theorem 3, we start with studying the regularized problem (4.2). Since $|S_\eta(u_\eta, v_\eta, x)| \leq C_S$, in light of Lemma 2.2, we can apply a slightly modified version of Theorem 2 to obtain global existence and boundedness for (4.2) under appropriate smallness conditions on the initial data.

Proposition 4.1 *Suppose $0 < \lambda' < \lambda_1$. Then there exists $\varepsilon_0 > 0$ with the following property:*

If $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\sigma}(\Omega)$ with $\sigma > 2$ are nonnegative and satisfy

$$\|u_0\|_{L^1(\Omega)} \leq \varepsilon \text{ and } \|\nabla v_0\|_{L^2(\Omega)} \leq \varepsilon \quad (4.6)$$

for some $\varepsilon \leq \varepsilon_0$, where ε_0 depends on Ω and λ' . Then the classical solution (u_η, v_η) of (4.2) exists globally and stays bounded. Moreover, there exists $M > 0$ depending on Ω and λ' such that

$$\|u_\eta - \bar{u}_0\|_{L^\infty(\Omega)} \leq M\varepsilon e^{-\lambda't}, \|v_\eta - \bar{u}_0\|_{L^\infty(\Omega)} \leq M\varepsilon e^{-\min\{\lambda', 1\}t}, \quad (4.7)$$

$$\|u_\eta\|_{L^\infty(\Omega)} \leq M, \|v_\eta\|_{L^\infty(\Omega)} \leq M, \quad (4.8)$$

$$\text{and} \quad \|u_\eta - v_\eta\|_{L^\infty(\Omega)} \leq 2M\varepsilon e^{-\min\{\lambda', 1\}t}, \quad (4.9)$$

$$\text{as well as} \quad \|\nabla v_\eta\|_{L^2(\Omega)} \leq M\varepsilon e^{-\lambda't} \text{ for all } t > 0. \quad (4.10)$$

Note that the above estimates are independent of η . Having obtained global existence and long time convergence for (4.2), we proceed to construct a solution of (4.1) upon letting $\eta \rightarrow 0$. This limit procedure needs some compactness properties of (u_η, v_η) , which are proven in the following lemmata.

Lemma 4.1 *There exists $C_1 > 0$ such that*

$$\int_0^\infty \int_\Omega |\nabla v_\eta|^2 \leq C_1, \quad (4.11)$$

$$\int_0^\infty \int_\Omega |\nabla u_\eta|^2 \leq C_1. \quad (4.12)$$

Proof The boundedness of $\int_0^\infty \int_\Omega |\nabla v_\eta|^2$ is an immediate consequence of (4.10). Next, we multiply the first equation of (4.2) by u_η to see that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega u_\eta^2 + \int_\Omega |\nabla u_\eta|^2 &= \int_\Omega u_\eta S_\eta(u_\eta, v_\eta, x) \nabla v_\eta \cdot \nabla u_\eta \\ &\leq \frac{1}{2} \int_\Omega |\nabla u_\eta|^2 + \frac{1}{2} C_S^2 M^2 \int_\Omega |\nabla v_\eta|^2 \end{aligned}$$

for all $t > 0$. Rearranging and integrating over $(0, \infty)$ imply

$$\int_0^\infty \int_\Omega |\nabla u_\eta|^2 \leq \int_\Omega u_0^2 + C_S^2 M^2 \int_0^\infty \int_\Omega |\nabla v_\eta|^2. \quad (4.13)$$

Finally, we can choose $C_1 > 0$ in an obvious way to establish (4.11) and (4.12). \square

The next lemma will ensure that $\frac{\partial}{\partial t}u_\eta$ and $\frac{\partial}{\partial t}v_\eta$ are bounded in $L^2([0, \infty); (W^{1,2}(\Omega))^*)$.

Lemma 4.2 *There exists $C_2 > 0$ such that*

$$\left\| \frac{\partial}{\partial t} u_\eta \right\|_{L^2([0, \infty); (W^{1,2}(\Omega))^*)} \leq C_2; \quad (4.14)$$

$$\left\| \frac{\partial}{\partial t} v_\eta \right\|_{L^2([0, \infty); (W^{1,2}(\Omega))^*)} \leq C_2. \quad (4.15)$$

Proof Let $\phi \in W^{1,2}(\Omega)$, and take ϕ as a test function in the first equation to obtain from (4.10)

$$\begin{aligned} \int_{\Omega} \frac{\partial}{\partial t} u_\eta \phi &= - \int_{\Omega} \nabla u_\eta \cdot \nabla \phi + \int_{\Omega} u_\eta S_\eta \nabla v_\eta \cdot \nabla \phi \\ &\leq \left(\int_{\Omega} |\nabla u_\eta|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla \phi|^2 \right)^{\frac{1}{2}} + MC_S \left(\int_{\Omega} |\nabla v_\eta|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla \phi|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\left(\int_{\Omega} |\nabla u_\eta|^2 \right)^{\frac{1}{2}} + MC_S \left(\int_{\Omega} |\nabla v_\eta|^2 \right)^{\frac{1}{2}} \right) \|\phi\|_{W^{1,2}(\Omega)}. \end{aligned}$$

This implies that

$$\left\| \frac{\partial}{\partial t} u_\eta(\cdot, t) \right\|_{(W^{1,2}(\Omega))^*} \leq \left(\int_{\Omega} |\nabla u_\eta|^2 \right)^{\frac{1}{2}} + MC_S \left(\int_{\Omega} |\nabla v_\eta|^2 \right)^{\frac{1}{2}}$$

for all $t > 0$, and hence

$$\int_0^\infty \left\| \frac{\partial}{\partial t} u_\eta(\cdot, t) \right\|_{(W^{1,2}(\Omega))^*}^2 \leq \int_0^\infty \int_{\Omega} |\nabla u_\eta|^2 + M^2 C_S^2 \int_0^\infty \int_{\Omega} |\nabla v_\eta|^2. \quad (4.16)$$

The right-hand side of (4.16) is bounded due to Lemma 4.1. Similarly, we multiply ϕ on both sides of the second equation to obtain

$$\begin{aligned} \int_{\Omega} \frac{\partial}{\partial t} v_\eta \phi &= - \int_{\Omega} \nabla v_\eta \cdot \nabla \phi - \int_{\Omega} v_\eta \phi + \int_{\Omega} u_\eta \phi \\ &\leq \left(\int_{\Omega} |\nabla v_\eta|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla \phi|^2 \right)^{\frac{1}{2}} + \int_{\Omega} (u_\eta - v_\eta) \phi \\ &\leq \left(\left(\int_{\Omega} |\nabla v_\eta|^2 \right)^{\frac{1}{2}} + \left(\int_{\Omega} |u_\eta - v_\eta|^2 \right)^{\frac{1}{2}} \right) \|\phi\|_{W^{1,2}(\Omega)}. \end{aligned}$$

Again, this fact together with (4.7) entails that

$$\left\| \frac{\partial}{\partial t} v_\eta(\cdot, t) \right\|_{(W^{1,2}(\Omega))^*} \leq \left(\int_{\Omega} |\nabla v_\eta|^2 \right)^{\frac{1}{2}} + \left(\int_{\Omega} |u_\eta - v_\eta|^2 \right)^{\frac{1}{2}}.$$

We integrate over $(0, \infty)$ to obtain

$$\int_0^\infty \left\| \frac{\partial}{\partial t} v_\eta(\cdot, t) \right\|_{(W^{1,2}(\Omega))^*}^2 \leq \int_0^\infty \int_{\Omega} |\nabla v_\eta|^2 + \int_0^\infty \int_{\Omega} |u_\eta - v_\eta|^2. \quad (4.17)$$

Collecting (4.9), (4.16), (4.17), (4.11) and (4.12), we see that if we choose

$$C_2 := (2 + M^2 C_S^2) C_1 + \frac{2M^2 |\Omega|}{\min\{\lambda_1, 1\}},$$

then (4.14) and (4.15) hold. \square

Now we can obtain the desired compactness properties of (u_η, v_η) to prove Theorem 3.

Proof of Theorem 3 First we note that both u_η and v_η are bounded in $L^2_{loc}([0, \infty); W^{1,2}(\Omega))$ according to Lemma 4.1 and (4.8). This fact together with Lemma 4.2 yields that the families $\{u_\eta\}$ and $\{v_\eta\}$ are strongly compact in $L^2_{loc}([0, \infty); L^2(\Omega))$ by invoking a version of the Aubin-Lions lemma [12]. We see that there exists $\{\eta_j\}_{j \in \mathbb{N}} \in (0, 1)$ satisfying $\eta_j \rightarrow 0$ as $j \rightarrow \infty$ and nonnegative functions $u, v \in L^2_{loc}([0, \infty); L^2(\Omega))$ such that

$$u_\eta \rightarrow u, v_\eta \rightarrow v \text{ in } L^2_{loc}([0, \infty); L^2(\Omega)) \text{ as } \eta = \eta_j \rightarrow 0. \quad (4.18)$$

According to (4.8), (4.11) and (4.12), we obtain the following properties of (u, v)

$$\begin{aligned} u_\eta &\overset{*}{\rightharpoonup} u, v_\eta \overset{*}{\rightharpoonup} v \text{ in } L^\infty((0, \infty) \times \Omega) \\ u_\eta &\rightarrow u, v_\eta \rightarrow v \text{ a.e. in } \Omega \times (0, \infty), \\ \text{and } \nabla u_\eta &\rightharpoonup \nabla u, \nabla v_\eta \rightharpoonup \nabla v \text{ in } L^2((0, T) \times \Omega). \end{aligned} \quad (4.19)$$

We see that (4.19) in combination with (4.3) implies

$$u_\eta S_\eta(u_\eta, v_\eta, x) \rightarrow u S(u, v, x) \text{ a.e. in } \Omega \times (0, \infty). \quad (4.20)$$

Choosing $\phi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$, we see that (u_η, v_η) also satisfies

$$-\int_0^\infty \int_\Omega u_\eta \phi_t - \int_\Omega u_0 \phi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla u_\eta \cdot \nabla \phi \quad (4.21)$$

$$\begin{aligned} &+ \int_0^\infty \int_\Omega u_\eta (S_\eta(u_\eta, v_\eta, x) \nabla v_\eta) \cdot \nabla \phi \\ -\int_0^\infty \int_\Omega v_\eta \phi_t - \int_\Omega v_0 \phi(\cdot, 0) &= -\int_0^\infty \int_\Omega \nabla v_\eta \cdot \nabla \phi \\ &- \int_0^\infty \int_\Omega v_\eta \phi + \int_0^\infty \int_\Omega u_\eta \phi. \end{aligned} \quad (4.22)$$

Here, (4.19) and (4.20) allow us to take $\eta = \eta_j \rightarrow 0$ in the above identities. Therefore (u, v) is weak solution of (1.1) in the natural sense. By standard parabolic theory [8, 9], (u, v) is in fact a classical solution of (4.1). Moreover, (4.19) enable us to take $\eta = \eta_j \rightarrow 0$ in (4.7) to derive (4.5). \square

5 Acknowledgement

The author would like to thank Professor Michael Winkler for guidance and useful comments on this paper, which largely improve the recent work.

References

- [1] X. Cao, S. Ishida, *Global-in-time bounded weak solutions to a degenerate quasilinear Keller-Segel system with rotation*, preprint.
- [2] L. Corrias, B. Perthame, H. Zaag, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math **72** (2004) 1-28.
- [3] L. Corrias, B. Perthame, *Asymptotic decay for the solutions of the parabolic-parabolic Keller-Segel chemotaxis system in critical spaces*, Mathematical and Computer Modelling **47** 755-764 (2008).

- [4] T. Hillen, K.J. Painter, *A user's guide to PDE models in a chemotaxis*, J. Math. Biology **58**, 183-217 (2009).
- [5] D. Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences I*, Jber. DMV **105** (3), 103-165.
- [6] D. Horstmann, M. Winkler, *Boundedness vs. blow-up in a chemotaxis system*, J. Differential Equations **215**, 52-107 (2005).
- [7] W. Jäger, S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Tran. Amer. Math. Soc. **329**, 819-824 (1992).
- [8] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type", American Mathematical Society, Providence, R.I., 1968.
- [9] T. Li, A. Suen, C. Xue, M. Winkler, *Small-data solutions in a chemotaxis system with rotation*, preprint.
- [10] E. F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399-415.
- [11] T. Nagai, T. Senba, K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. **3**, 411-433, (1997).
- [12] R. Temam, Navier-Stokes Equations. Theory and Numerical Analysis, Stud. Math. Appl. **2**, North-Hlland, Amsterdam, 1997.
- [13] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, R. E. Goldstein, *Bacterial swimming and oxygen transport near contact lines*, Proc. Nat. Acad. Sci, **102** (2005), 2277-2282.
- [14] C. Xue, H. G. Othmer, *Multiscale models of taxis-driven patterning in bacterial population*, SIAM J. Appl. Math. **70** (2009), 133-167.
- [15] M. A. Herrero, J. J. L. Velázquez, *A blow-up mechanism for a chemotaxis model*, Ann. Scuola Normale Superiore **24**, 633-683 (1997).
- [16] T. Nagai, *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains*, J. Inequal. Appl. **6**, 37-55 (2001).
- [17] M. Winkler, *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model*, J. Differential Equations **248**, 2889-2905 (2010).
- [18] M. Winkler, *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system*, J. Math. Pures Appl. **100** 748-767 (2013).